HANKEL TYPE CONVOLUTION EQUATIONS IN DISTRIBUTION SPACE

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Abstract:

In this paper we study Hankel type convolution equations in distribution spaces. Solvability conditions for Hankel type convolution equations are obtained. We have also investigated hypoelliptic Hankel-type convolution equations.

Key Words: Hankel type convolution equations, distributions, Bessel type functions.

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1. Introduction: The Hankel type transformation is usually defined by

$$h_{\alpha,\beta}\,(\phi)\,(x) = \int_0^\infty (xt)^{\alpha+\beta}\, J_{\alpha-\beta}\,(xt)\,\phi(t)\,dt,\ \ x \in I = (0,\infty),$$

where $J_{\alpha-\beta}$ denotes the Bessel type function of the first kind and order $(\alpha-\beta)$. Throughout this paper $(\alpha-\beta)$ always will be greater than $-\frac{1}{2}$, and will denote by I the real interval $(0,\infty)$.

Following [25,26, and 27], we introduce the space $\mathcal{H}_{\alpha,\beta}$ as the space of all those complex valued and smooth functions ϕ defined on I such that, for any $m\,,k\ \in\ \mathbb{N}$,

$$\rho_{m,k}^{\alpha,\beta}(\phi) = \operatorname*{Sup}_{x\in(0,\infty)} \left| x^m \left(\frac{1}{x}\,D\right)^k \left[x^{2\beta-1}\phi(x)\right]\right| < \infty\,.$$

The space $\mathcal{H}_{\alpha,\beta}$ is Frechet when it is endowed with the topology generated by the family $\left\{\rho_{m,k}^{\alpha,\beta}\right\}_{m,k\,\in\,\mathbb{N}}$ of seminorms. Following [25, Lemma 8], it can be easily established that $h_{\alpha,\beta}$ is an automorphism of $\mathcal{H}_{\alpha,\beta}$. The Hankel type transformation is defined on $\mathcal{H}'_{\alpha,\beta}$, the dual space of $\mathcal{H}_{\alpha,\beta}$, as the adjoint of the $h_{\alpha,\beta}-$ transformation of $\mathcal{H}_{\alpha,\beta}$, and it is denoted by $h'_{\alpha,\beta}$. More recently Waphare and Gunjal [24] have studied $h_{\alpha,\beta}$ on new spaces of functions and distributions. Now we define the spaces $\chi_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ as follows:

A complex valued and smooth function ϕ defined on I is in $\chi_{\alpha,\beta}$ if and only if, for every $m\,,k\ \in\ \mathbb{N}$,

$$\eta_{m,k}\,(\phi) = \lim_{n\to\infty} \left| e^{mx} \left(\frac{1}{x}\,D\right)^k \left(x^{2\beta-1}\,\phi(x)\right)\right| < \infty\,.$$

$\chi_{\alpha,\beta}$ is equipped with the topology associated to the system $\left\{\eta_{m,k}^{\alpha,\beta}\right\}_{m,k\,\in\mathbb{N}}$ of seminorms. Thus $\chi_{\alpha,\beta}$ is a Frechet space.

The space $Q_{\alpha,\beta}$ is constituted by all those complex valued functions Φ satisfying the following two conditions:

(i) $s^{2\beta-1}\,\Phi\,(s)$ is an even entire function, and

(ii) for every $m\,,k\ \in\ \mathbb{N}$

$$\lambda_{m,k}^{\alpha,\beta}\,(\Phi) = \operatorname*{Sup}_{|Ims|\le k}\,(1+|s|^2)^m\left|s^{2\beta-1}\Phi(s)\right| < \infty.$$

$Q_{\alpha,\beta}$ is a Frechet space when we consider the topology generated by the family of seminorms $\left\{\lambda_{m,k}^{\alpha,\beta}\right\}_{m,k\,\in\mathbb{N}}$ on $Q_{\alpha,\beta}$.

In [24] it is established that $h_{\alpha,\beta}$ is a homeomorphism from $\chi_{\alpha,\beta}$ onto $Q_{\alpha,\beta}$. Moreover, $h_{\alpha,\beta}$ coincides with its inverse. The Hankel type transform is defined on the dual spaces $\chi_{\alpha,\beta}^{'}$ and $Q_{\alpha,\beta}^{'}$ as the adjoint of the $h_{\alpha,\beta}$ transformation and it is also denoted by $h_{\alpha,\beta}^{'}$.

The convolution for a Hankel type transformation closely connected with $h_{\alpha,\beta}$ was investigated by Hirschman [9] and Haimo [8] and Cholewinski [5]. A simple manipulation in the convolution considered by the above authors allows us to obtain the convolution for $h_{\alpha,\beta}$ that will denoted by # and is defined as follows: For every measurable function ϕ and ψ on I such that $x^{2\alpha}\,\phi$ and $x^{2\alpha}\,\psi$ are absolutely integrable on I, the convolution $\phi\,\#\,\psi$ of ϕ and ψ is given by

$$(\phi\#\psi)\,(x) = \int_0^\infty \phi(y)\,(\tau_x\psi)\,(y)\,dy,\;\; x\,\in I,$$

where

$$(\tau_x\psi)\,(y) = \int_0^\infty D_{\alpha,\beta}\,(x,y,z)\,\psi\,(z)\,dz,\;\; x,y\,\in I\;\;\text{and}$$

$$D_{\alpha,\beta}\,(x,y,z) = \int_0^\infty t^{2\beta-1}(xt)^{\alpha+\beta}J_{\alpha-\beta}(xt)\,(yt)^{\alpha+\beta}\,J_{\alpha-\beta}\,(yt)\,(zt)^{\alpha+\beta}\,J_{\alpha-\beta}\,(zt)\,dt,$$

$x,y,z\;\in I.$

The study of the $\#-$convolution in distribution spaces was started by de Sousa-Pinto [19]. In a series of papers, Betancor and Marrero [2,3,4,22,23] and [12] have investigated the Hankel convolution on the Zemanian spaces. Also Betencor and Gonzalez [1] studied the generalized Hankel convolution. Recently, Waphare and Gunjal [24] defined the # convolution on distributions of exponential growth.

In this paper we analyze Hankel type convolution equations. Solvability conditions for the # convolution equations in $\mathcal{H}_{\alpha,\beta}^{'}$ and $\chi_{\alpha,\beta}^{'}$ are investigated in Section 2. Also in Section 3 we study hypoelliptic Hankel type convolution equations in $\mathcal{H}_{\alpha,\beta}^{'}$ and $\chi_{\alpha,\beta}^{'}$. Throughout this paper M will always denote a suitable positive constant not necessarily the same in each occurrence.

2. Solvability of Hankel type convolution equations of distribution:

In this section, inspired by the papers of Sznajder and Zietezny [17,18] and Pahk and Sohn [15], we obtain necessary and sufficient conditions to solve Hankel type convolution equations in $\mathcal{H}'_{\alpha,\beta}$ and $\chi'_{\alpha,\beta}$. Marrero and Betencor [12] studied the Hankel type convolution operators on $\mathcal{H}'_{\alpha,\beta}$. They introduced, for every $m \in Z$, the space $O_{\alpha,\beta,m,\#}$ constituted by all those complex valued and smooth functions ϕ defined on I such that, for every $k \in \mathbb{N}$,

$$\delta_k^{\alpha,\beta,m}(\phi) = \underset{x \in I}{Sup}\left|(1+x^2)^m\, x^{2\beta-1}\, \Delta_{\alpha,\beta}^k\, \phi(x)\right| < \infty,$$

where $\Delta_{\alpha,\beta}$ denotes the Bessel type operator $x^{2\beta-1} D\, x^{4\alpha}\, D\, x^{2\beta-1}$.

We define $\bar{O}_{\alpha,\beta,m,\#}$ as the closure of $\mathcal{H}_{\alpha,\beta}$ in $O_{\alpha,\beta,m,\#}$.

Note that $\bar{O}_{\alpha,\beta,m,\#} \supset \bar{O}_{\alpha,\beta,m+1,\#}$ for each $m \in Z$. The space $\bigcup_{m \in z} \bar{O}_{\alpha,\beta,m,\#}$ is denoted by $\bar{O}_{\alpha,\beta,\#}$. The Hankel type convolution operators of $\mathcal{H}'_{\alpha,\beta}$ are the elements of $\bar{O}'_{\alpha,\beta,\#}$, the dual space of $\bar{O}_{\alpha,\beta,\#}$ [3]. Characterizations of $\bar{O}'_{\alpha,\beta,\#}$, were obtained in Proposition 4.2 [12]. Following [4], we can establish the following result:

Proposition 2.1: For $S \in \bar{O}'_{\alpha,\beta,\#}$, the following conditions are equivalent

(i) To every $k \in \mathbb{N}$ there correspond $m, n \in \mathbb{N}$ and a positive constant M such that

$$\max_{o \le \ell \le m} Sup \left\{\left|\left(\frac{1}{t}D\right)^{\ell}\left[t^{2\beta-1}(h'_{\alpha,\beta}S)(t)\right]\right| : t \in I,\ |x-t| \le (1+x^2)^{-k}\right\}$$

$\ge (1+x^2)^{-n}$, whenever $x \in I,\ x > M$.

(ii) If $T \in \bar{O}'_{\alpha,\beta,\#}$ and $S\#T \in \mathcal{H}_{\alpha,\beta}$, then $T \in \mathcal{H}_{\alpha,\beta}$.

If $S \in \bar{O}'_{\alpha,\beta,\#}$, the existence of solution for the convolution equation

$$u \,\#\, S = v\,, \tag{2.1}$$

for every $v \in \mathcal{H}'_{\alpha,\beta}$; implies conditions (i) and (ii) in Proposition 2.1.

Proposition 2.2: Let $S \in \bar{O}'_{\alpha,\beta,\#}$. If $\mathcal{H}'_{\alpha,\beta} \,\#\, S = \mathcal{H}'_{\alpha,\beta}$, then conditions (i) and (ii) in Proposition 2.1 hold.

Proof: It is enough to see that (ii) holds when $\mathcal{H}'_{\alpha,\beta} \,\#\, S = \mathcal{H}'_{\alpha,\beta}$.

Note that the mapping

$$F : \mathcal{H}'_{\alpha,\beta} \to \mathcal{H}'_{\alpha,\beta} = \mathcal{H}'_{\alpha,\beta} \# S$$

$$u \to u\#S$$

is the transpose of the mapping

$$G: \mathcal{H}_{\alpha,\beta} \rightarrow \mathcal{H}_{\alpha,\beta} \subset S \,\#\, \mathcal{H}_{\alpha,\beta}$$
$$\phi \rightarrow S \,\#\phi\,.$$

Then by involving [6, Corollary, p. 92] the mapping G is an isomorphism.

In particular, the mapping $G^{-1}: S \,\#\, \mathcal{H}_{\alpha,\beta} \rightarrow \mathcal{H}_{\alpha,\beta}$ is continuous.

Assume now that $T \in \bar{O}'_{\alpha,\beta,\#}$ is such that $T \,\#\, S \in \mathcal{H}_{\alpha,\beta}$. Let $(\phi_k)_{k=1}^{\infty}$ be a sequence of smooth functions such that the following three conditions are satisfied.

(i) $c_{\alpha,\beta}^{-1} \int_0^{\infty} x^{2\alpha} \phi_k(x)\, dx = 1$, where $c_{\alpha,\beta} = 2^{\alpha-\beta}\Gamma(3\alpha+\beta)$,

(ii) $0 \le \phi_k(x),\ x \in I$,

(iii) $\phi_k(x) = 0,\ x \notin (1/(k+1),\ 1/k)$, for every $k \in \mathbb{N}$.

According to [3,p.1148], for each $\phi \in \mathcal{H}_{\alpha,\beta}$,

$$\phi_k \,\#\, \psi \rightarrow \psi\,,\ as\ k \rightarrow \infty,\ in\ \mathcal{H}_{\alpha,\beta}\,. \qquad (2.2)$$

Moreover, by involving [12, Proposition 4.7], we can write

$$S \,\#\, (T\#\phi_k) = (S\#T) \,\#\, \phi_k = (T\#S)\#\, \phi_k\,, \text{for every } k \in \mathbb{N}\,.$$

Since $T \,\#\, \phi_k \in \mathcal{H}_{\alpha,\beta}$, $k \in \mathbb{N}$, by taking into account that $G^{-1}$is continuous and by (2.2) and (2.3), we conclude that $(T\#\phi_k)_{k=1}^{\infty}$ converges in $\mathcal{H}_{\alpha,\beta}$. Also by (2.2) again $T \,\#\, \phi_k \rightarrow T$, as $k \rightarrow \infty$, in $\mathcal{H}'_{\alpha,\beta}$. When we consider in $\mathcal{H}'_{\alpha,\beta}$ the weak* (or the strong) topology. Hence $T \in \mathcal{H}_{\alpha,\beta}$. Thus proof is completed.

Waphare and Gunjal [24] have defined the Hankel type convolution of distributions of exponential growth. We introduced [24] a subspace $\chi'_{\alpha,\beta,\#}$ of $\chi'_{\alpha,\beta}$ consisting of $S \in \chi'_{\alpha,\beta}$ such that $S\#\psi \in \chi_{\alpha,\beta}$ for every $\psi \in \chi_{\alpha,\beta}$.

In the following we establish a condition that $S \in \chi'_{\alpha,\beta,\#}$ satisfies when the equation (2.1) admits a solution for every $v \in \chi'_{\alpha,\beta}$.

Proposition 2.3: Let $S \in \chi'_{\alpha,\beta,\#}$. If $\chi'_{\alpha,\beta} \,\#\, S = \chi'_{\alpha,\beta}$, then S verifies the following property. $T \in \chi_{\alpha,\beta}$ provided that $T \in \chi'_{\alpha,\beta,\#}$ and $T\#\, S \in \chi_{\alpha,\beta}$.

Proof: This result can be proved in a similar way to Proposition 2.2. It is enough to see that if $(\psi_k)_{k=1}^{\infty}$ is a sequence of smooth functions verifying the three conditions listed in the proof of Proposition 2.2 then, for every $\psi \in \chi_{\alpha,\beta}$,

$$\psi \,\#\, \phi_k \rightarrow \psi,\ as\ k \rightarrow \infty,\ in\ \chi_{\alpha,\beta}\,, \qquad (2.4)$$

By virtue of [24] and by the interchange formula [9, Theorem 2d] to show (2.4) it is equivalent to see that, for every $\Psi \in Q_{\alpha,\beta}$,

$$s^{2\beta-1}h_{\alpha,\beta}(\phi_k)\,\Psi \to \Psi \text{ as } k\to\infty \text{, in } Q_{\alpha,\beta}. \quad (2.5)$$

We now prove (2.5). Let $(\phi_k)_{k=1}^{\infty}$ be a sequence in the proof of Proposition 2.2 and Let $\Psi \in Q_{\alpha,\beta}$. Since $\int_0^\infty t^{2\alpha}\,\phi_k(t)\,dt = c_{\alpha,\beta}$, for every $k \in \mathbb{N}$, where $c_{\alpha,\beta} = 2^{\alpha-\beta}\Gamma(3\alpha+\beta)$, we can write

$$s^{2\beta-1}h_{\alpha,\beta}(\phi_k)(s) - 1 = \int_0^\infty (st)^{-(\alpha-\beta)}J_{\alpha-\beta}(st)\,t^{2\alpha}\,\phi_k(t)\,dt - 1$$

$$= \int_{1/(k+1)}^{1/k} \left[(st)^{-(\alpha-\beta)}J_{\alpha-\beta}(st) - \frac{1}{c_{\alpha,\beta}}\right] t^{2\alpha}\,\phi_k(t)\,dt\,,$$

for every $k \in \mathbb{N}$ and $s \in \mathbb{C}$.

Let K be a compact subset of $\mathbb{C}$, and let $\epsilon > 0$. There exists $t_0 > 0$ such that

$$\left|(st)^{-(\alpha-\beta)}J_{\alpha-\beta}(st) - 1/c_{\alpha,\beta}\right| < \epsilon,$$

for each $0 < t < t_0$ and $s \in K$. Hence we can find $k_0 \in \mathbb{N}$ such that every $k \geq k_0$ and $s \in K$,

$$\left|s^{2\beta-1}h_{\alpha,\beta}(\phi_k)(s) - 1\right| \leq \int_{1/(k+1)}^{1/k} \left|(st)^{-(\alpha-\beta)}J_{\alpha-\beta}(st) - 1/C_{\alpha,\beta}\right| t^{2\alpha}\,\phi_k(t)\,dt$$

$$< \epsilon\, c_{\alpha,\beta}\,.$$

Moreover, from [11, Lemma 4], we deduce

$$\left|s^{2\beta-1}h_{\alpha,\beta}(\phi_k)(s) - 1\right| \leq \int_0^\infty \left(\left|(st)^{-(\alpha-\beta)}J_{\alpha-\beta}(st)\right| + 1\right) t^{2\alpha}\,\phi_k(t)\,dt$$

$$\leq M\,e^{|Ims|}\int_0^\infty t^{2\alpha}\,\phi_k(t)\,dt$$

$$= M\,e^{|Ims|}\,, \text{ for every } k \in \mathbb{N} \text{ and } s \in \mathbb{C}\,.$$

Hence, for each $m \in \mathbb{N}$ there exists a $a > 0$ such that

$$\frac{1}{1+|s|^2}\left|s^{2\beta-1}h_{\alpha,\beta}(\phi_k) - 1\right| < \epsilon\,, \text{ for every } k \geq k_0 \text{ and } |Ims| \leq m.$$

Further let $m, n \in \mathbb{N}$. We have, for every $\Psi \in Q_{\alpha,\beta}$,

$$w_{n,m}^{\alpha,\beta}\left(\Psi(s)\left[s^{2\beta-1}h_{\alpha,\beta}(\phi_k)(s) - 1\right]\right)$$

$$\leq \underset{|Ims|\leq m}{\text{Sup}} (1+|s|^2)^{n+1} \left| s^{2\beta-1}\,\Psi(s) \left| \underset{|Ims|}{\text{Sup}} \frac{1}{1+|s|^2} \right| s^{2\beta-1}\, h_{\alpha,\beta}(\phi_k)\,(s) - 1 \right| \to 0,$$

as $k \to \infty$. Thus (2.5) is proved. Thus proof is completed.

We now give a condition for $S \in \chi'_{\alpha,\beta,\#}$ that implies the solvability of equation (2.1) for every $v \in \chi'_{\alpha,\beta}$.

Proposition 2.4: Let $S \in \chi'_{\alpha,\beta,\#}$. If there exist N, τ, C positive constants such that

$$Sup_{s\in\mathbb{C},|s|\leq r}\left|(\xi+s)^{2\beta-1}h'_{\alpha,\beta}\,(S)\,(\xi+s)\right| \geq \frac{C}{(1+|\xi|^2)^N}, \quad \xi \in \mathbb{R}, \qquad (2.6)$$

then $\chi'_{\alpha,\beta}\,\#\,S = \chi'_{\alpha,\beta}$.

Proof: By $[6, Corollary,\ p.92]$, we see that $\chi'_{\alpha,\beta} = \chi'_{\alpha,\beta}\,\#\,S$, it is sufficient to prove that the linear mapping

$$G: \chi_{\alpha,\beta} \to S\,\#\,\chi_{\alpha,\beta} \subset \chi_{\alpha,\beta}$$
$$\psi \to S\#\psi$$

is a homeomorphism.

Note firstly that G is continuous mapping. In effect, by invoking [24], we obtain

$$G\,(\psi) = S\#\psi = h_{\alpha,\beta}\left(S^{2\beta-1}\,h'_{\alpha,\beta}\,(S)\,h_{\alpha,\beta}\,(\psi)\right), \; for\ every\ \psi \in \chi_{\alpha,\beta}. \quad (2.7)$$

Since $s^{2\beta-1}\,h'_{\alpha,\beta}\,(S)$ is a continuous multiplier from $Q_{\alpha,\beta}$ into itself (see [24]) and from [24] it infers that G is continuous.

Moreover from (2.7), we can deduce that G is one-to-one. Infact, if $\psi \in \chi_{\alpha,\beta}$ being $G\,(\psi) = 0$ then $s^{2\beta-1}\,h'_{\alpha,\beta}\,(S)\;h_{\alpha,\beta}\,(\psi) = 0$. Since $S \neq 0$, $h_{\alpha,\beta}\,(\psi) = 0$ and hence $\psi = 0$. To complete the proof, we have to prove that the mapping

$$G^{-1}: S\,\#\,\chi_{\alpha,\beta} \to \chi_{\alpha,\beta}$$
$$S\#\,\psi \to \psi$$

is continuous, or equivalently, by [24], we have to see that the mapping

$$F: s^{2\beta-1}\,h'_{\alpha,\beta}\,(S)\;Q_{\alpha,\beta} \to Q_{\alpha,\beta}$$
$$s^{2\beta-1}\;h'_{\alpha,\beta}\,(S)\,\Phi \to \Phi$$

is continuous. Let $\Phi \in Q_{\alpha,\beta}$ and define $\Psi = s^{2\beta-1}\,h'_{\alpha,\beta}\,(S)\Phi$. Let $k \in \mathbb{N}$. By invoking lemma of Hormander [10, Lemma 3.2], we obtain

$$\left|s^{2\beta-1}\Phi\,(s)\right| \leq \sup_{|z-s|<4(k+r)} \left|z^{2\beta-1}\,h'_{\alpha,\beta}(S)(z)\,z^{2\beta-1}\,\Phi\,(z)\right|$$

$$\times \frac{\sup_{|z-s|<4(k+r)} \left|z^{2\beta-1}\ h'_{\alpha,\beta}\,(S)(z)\right|}{\left[\sup_{|z-s|<k+r} \left|z^{2\beta-1}\ h'_{\alpha,\beta}\,(S)(z)\right|\right]^2}\ ,\ s \in \mathbb{C}\,.$$

Also, according to (2.6), one has

$$\sup_{|z-s|<k+r} \left|z^{2\beta-1}\ h'_{\alpha,\beta}\,(S)(z)\right| = \sup_{|z|<k+r} \left|(s+z)^{2\beta-1}\ h'_{\alpha,\beta}\,(S)\,(s+z)\right|$$

$$\geq \sup_{|z|<r} \left|(Res+z)^{2\beta-1}\ h'_{\alpha,\beta}\,(S)\,(Res+z)\right|$$

$$\geq \frac{c}{(1+|Res|^2)^N} \qquad (2.9)$$

$$\geq \frac{C}{(1+|s|^2)^N}\ , \qquad |Ims|\ \leq k\,.$$

Moreover by [24], there exists $n \in \mathbb{N}$ such that

$$\sup_{|Imz|\leq 5k+4r} (1+|z|^2)^{-n}\left|z^{2\beta-1}\ h'_{\alpha,\beta}\,(s)\,(z)\right| < \infty\,.$$

Then

$$\sup_{|z-s|<4(k+r)} \left|z^{2\beta-1}\ h'_{\alpha,\beta}\,(s)\,(z)\right| = \sup_{|z|<4(k+r)} \left|(s+z)^{2\beta-1}\,h'_{\alpha,\beta}\,(s)\,(s+tz)\right|$$

$$\leq M \sup_{|z-s|<4(k+r)} (1+|s+z|^2)^n$$

$$\leq\ M\,(1+|s|^2)^n,\ |Ims|\ \leq k \qquad (2.10)$$

Hence from (2.8), (2.9) and (2.10), we conclude that

$$\left|s^{2\beta-1}\,\Phi(s)\right|\ \leq M\,(1+|s|^2)^{n+2N}$$

$$\times\ \sup_{|z|<4(k+r)}\left|(z+s)^{2\beta-1}\,\Psi(z+s)\right|\ ,|Ims|\ \leq k. \qquad (2.11)$$

Now let $m\ \in\ \mathbb{N}$. By (2.11) one has

$$\sup_{|Ims|\leq k} (1+|s|^2)^m \left|s^{2\beta-1}\,\Phi\,(s)\right|$$

$$\leq M \sup_{|Ims|\leq k} (1+|s|^2)^{n+2N+m} \sup_{|z|<4(k+r)} \left|(z+s)^{2\beta-1}\,\Psi(z+s)\right|$$

$$\leq M \sup_{|Ims|\leq k}\ .\ \sup_{|z|<4(k+r)} (1+|z+s|^2)^{n+2N+m} \left|(z+s)^{2\beta-1}\Psi(z+s)\right|$$

$$\leq M \sup_{|Ims|\ \leq\ 5k+4r} (1+|s|^2)^{n+2N+m} \left|s^{2\beta-1}\,\Psi(s)\right|.$$

Thus we prove that F is continuous, and we conclude that G is a homeomorphism. Thus proof is completed.

3. Hypoelliptic Hankel type convolution equations:

Sampson and Zielezny [16], Zielezny [28] and [29] and Pahk [13] and [14], amongst others, have investigated hypoelliptic (usual) convolution equations in certain spaces of generalized functions.

In this section we investigate hypoelliptic conditions for the Hankel type convolution equations in $\mathcal{H}'_{\alpha,\beta}$ and $\chi'_{\alpha,\beta}$.

Let $S \in \bar{O}'_{\alpha,\beta,\#}$. We say that S (or the Hankel type convolution equation $u\#s = v$) is hypoelliptic in $\mathcal{H}'_{\alpha,\beta}$ if all solutions $u \in \mathcal{H}'_{\alpha,\beta}$ of $u\,\#S = v$ are in $\bar{O}_{\alpha,\beta,\#}$ whenever $v \in \bar{O}_{\alpha,\beta,\#}$.

Conversely $v \in \bar{O}_{\alpha,\beta,\#}$ provided that the equation $u\#S = v$ admits a solution $u \in \bar{O}_{\alpha,\beta,\#}$.

Proposition 3.1: If $f \in \bar{O}_{\alpha,\beta,\#}$ and $S \in \bar{O}'_{\alpha,\beta,\#}$, then $f\,\#\,S \in \bar{O}_{\alpha,\beta,\#}$.

Proof: A simple modification in the proof of [12, Proposition 4.2] allows us to see that, for every $m \in \mathbb{N}$, there exist $k = k(m)$ and continuous functions f_p on I, $0 \le p \le k$, such that

$$S = \sum_{p=0}^{k} \Delta^p_{\alpha,\beta}\, f_p \quad , and$$

$(1 + x^2)^m\, x^{2\beta-1}\, f_p$ is bounded on I, $0 \le p \le k$.

Claim 1: $l \in \mathbb{Z}$, and let $f \in \bar{O}_{\alpha,\beta,l,\#}$. If $S \in \bar{O}'_{\alpha,\beta,\#}$, then

$$f\#S = \sum_{p=0}^{k} \Delta^p_{\alpha,\beta} \left(f\#f_p\right)$$

where $\left(f_p\right)_{p=0}^{k}$ is a family of continuous functions on $(0,\infty)$ such that

$$S = \sum_{p=0}^{k} \Delta^p_{\alpha,\beta}\, f_p \qquad (3.1)$$

and $(1 + x^2)^m\, x^{2\beta-1}\, f_p$ is bounded on I, for every $p = 0,1,\ldots\ldots,k$, and being $m > |l| + 3\alpha + \beta$.

Proof of claim 1: Let $\phi \in \mathcal{H}_{\alpha,\beta}$. By (3.1) we can write

$$\langle f\#S, \quad \psi\rangle = \langle f;\ S\#\psi = \int_0^\infty f(x) \sum_{p=0}^{k} \int_0^\infty f_p\,(y)\left(\tau_x\, \Delta^p_{\alpha,\beta}\psi\right)\rangle\ (y)dy\,dx$$

$$= \sum_{p=0}^{k} \int_0^\infty \left(\Delta^p_{\alpha,\beta}\psi\right)(x) \int_0^\infty f_p\,(y)\,(\tau_x\, f)\,(y)\,dy\,dx\,.$$

Since $m > |l| + 3\alpha + \beta$, it follows for every $p = 0, 1, \ldots\ldots, k$,

$$\int_0^\infty \int_0^\infty |f_p(y)|\,|f(x)| \int_0^\infty D_{\alpha,\beta}(x,y,z)\,\left|\Delta^p_{\alpha,\beta,z}\,\psi(z)\right|\,dz\,dx\,dy$$

$$= \int_0^\infty \int_0^\infty |f_p(y)|\,\left|\Delta^p_{\alpha,\beta,z}\,\psi(z)\right| . \int_{|z-y|}^{z+y} D_{\alpha,\beta}(x,y,z)\,x^{2\alpha}\,x^{2\beta-1}\,|f(x)|\,dx\,dy\,dz$$

$$\le M \int_0^\infty \int_0^\infty |f_p(y)|\,\left|\Delta^p_{\alpha,\beta,z}\,\psi(z)\right| (1+(z+y)^2)^{|l|}\,(zy)^{2\alpha}\,dz\,dy$$

$$\le M \int_0^\infty y^{2\alpha}\,(1+y^2)^{|l|}\,|f_p(y)|\,dy \,. \int_0^\infty z^{2\alpha}\,(1+z^2)^{|l|}\,\left|\Delta^p_{\alpha,\beta,z}\,\psi(z)\right|\,dz < \infty,$$

and the interchange in the order of integrations is justified.

Thus proof of claim 1 is completed.

Claim 2: Let $l \in z$. If g is a continuous function on I such that $(1+x^2)^a\,x^{2\beta-1}\,g(x)$ is bounded on I, for some $a > |l| + 3\alpha + \beta$, and $f \in \bar{O}_{\alpha,\beta,l,\#}$ then $f\#g \in \bar{O}_{\alpha,\beta,\#}$.

Proof: Let $b \in \mathbb{N}$. Following [1, Lemma 3.1], we can see that the operators τ_x and $\Delta_{\alpha,\beta}$ commute on $\bar{O}_{\alpha,\beta,\#}$ for each $x \in I$, we can write

$$\Delta^b_{\alpha,\beta}\,(f\#g)\,(x) = \int_0^\infty g(y)\,\tau_x\left(\Delta^b_{\alpha,\beta}\,f\right)(y)\,dy\,,\quad x \in I.$$

For every $x, y \in I$ one has

$$\left|\tau_x\left(\Delta^b_{\alpha,\beta}\,f\right)(y)\right| \le \int_{|x-y|}^{x+y} D_{\alpha,\beta}(x,y,z)\,\left|\left(\Delta^b_{\alpha,\beta}\,f\right)(z)\right|\,dz$$

$$\le M\,(1+(x+y)^2)^{|l|}\,(xy)^{2\alpha}$$

$$\le M\,(xy)^{2\alpha}\,(1+x^2)^{|l|}\,(1+y^2)^{|l|}.$$

Hence we obtain that

$$\left|\Delta^b_{\alpha,\beta}\,(f\#g)\,(x)\right| \le \int_0^\infty |g(y)|\,\left|\left(\tau_x\Delta^b_{\alpha,\beta}\,f\right)(y)\right|\,dy$$

$$\le M\,x^{2\alpha}\,(1+x^2)^{|l|}\,\underset{z\in I}{Sup}\,(1+z^2)^l\,z^{2\beta-1}\,\left|\Delta^b_{\alpha,\beta}\,f(z)\right|$$

$$\times \int_0^\infty y^{2\alpha}(1+y^2)^{|l|}\,|g(y)|\,dy,\; x \in I\,. \qquad (3.2)$$

Then $f\#g \in O_\alpha - |l|, \#$.

Moreover, $f\#g \in \bar{O}_{\alpha,\beta,-|l|,\#}$. In effect, if $(\psi_n)_{n=0}^{\infty} \subset \mathcal{H}_{\alpha,\beta}$ and $\psi_n \to f$, as $n \to \infty$ in $O_{\alpha,\beta,l,\#,}$ then from (3.2) we can infer that $\psi_n\#g \to f\#g,\ as\ n \to \infty$ in $O_{\alpha,\beta,-|l|,\#}$. Also according to [22], there exists an $s \in Z$ such that $\psi_n\#g \in \bar{O}_{\alpha,\beta,s,\#}$ for every $n \in \mathbb{N}$. Hence as $\bar{O}_{\alpha,\beta,\#}$ is complete [22], $f\#g \in \bar{O}_{\alpha,\beta,\#}$.

Now, by taking into account that, for every $\psi \in \mathcal{H}_{\alpha,\beta}$ and $f \in \bar{O}_{\alpha,\beta,\#}$,

$$\int_0^\infty f(x)\,\Delta_{\alpha,\beta}\,\psi(x)\,dx = \int_0^\infty \Delta_{\alpha,\beta}\,f(x)\,\psi(x)\,dx\,,$$

Thus from Claim 1 and Claim 2 we conclude that $f\#S \in \bar{O}_{\alpha,\beta,\#}$.

Thus proof is completed.

We say that $S \in \bar{O}'_{\alpha,\beta,\#}$ has the property (HE) if and only if there exist $B, C > 0$ such that $|h'_{\alpha,\beta}(S)(y)| \geq y^{-B}$ for every $y \geq C$. We now prove that the property (HE) is a necessary and sufficient condition in order that $S \in \bar{O}'_{\alpha,\beta,\#}$ is hypoelliptic in $\mathcal{H}'_{\alpha,\beta}$.

The following result will allow us to prove the necessity of the condition (HE).

Proposition 3.2: Assume that $\xi_1 > 1, \xi_j - \xi_{j-1} > 1$ for every $j = 2,3,\ldots,$ and $(a_j)_{j=1}^{\infty} \subset \mathbb{C}$ such that $|a_j| = O\left(\xi_j^{\gamma}\right),\ as\ j \to \infty$, for some $\gamma > 0$.

Denote by $\delta_{\alpha-\beta}$ the element of $\mathcal{H}'_{\alpha,\beta}$ defined by

$$\langle \delta_{\alpha-\beta}, \psi \rangle = c_{\alpha,\beta} \lim_{x\to 0+} x^{2\beta-1}\,\psi(x)\,,\ \psi \in \mathcal{H}_{\alpha,\beta}\,.,$$

being$c_{\alpha,\beta} = 2^{\alpha-\beta}\Gamma(3\alpha+\beta)$.

Then

$$\sum_{j=1}^{\infty} a_j\,\tau_{\xi_j}\,\delta_{\alpha-\beta} \in \mathcal{H}'_{\alpha,\beta}\,.$$

Moreover,if

$T = h'_{\alpha,\beta}\left(\sum_{j=1}^{\infty} a_j\,\tau_{\xi_j}\,\delta_{\alpha-\beta}\right)$, then $T \in \bar{O}_{\alpha,\beta,\#}$ if and only if $|a_j| = O\left(\xi_j^{-\nu}\right)$ as $j \to \infty$, for each $\nu \in \mathbb{N}$.

Proof: The series

$$\sum_{j=1}^{\infty} a_j\,\tau_{\xi_j}\,\delta_{\alpha-\beta}$$

converges in $\mathcal{H}'_{\alpha,\beta}$, when we consider in $\mathcal{H}'_{\alpha,\beta}$ the weak* topology. In effect for every $\psi \in \mathcal{H}_{\alpha,\beta}$ and $\xi \in I$ according to [4, (2.1)], one has

$$\langle \tau_\xi \ \delta_{\alpha-\beta}, \psi \rangle = c_{\alpha,\beta} \lim_{x \to 0+} x^{2\beta-1} \left(\tau_\xi \ \phi\right)(x)$$

$$= c_{\alpha,\beta} \lim_{x \to 0+} h_{\alpha,\beta} \left[(xt)^{-(\alpha-\beta)} J_{\alpha-\beta}\,(xt)\, h_{\alpha,\beta}\,(\psi)(t)\right](\xi) \qquad (3.3)$$

$$= \phi\, \psi\,(\xi)\,.$$

Hence for each $n \in \mathbb{N}$,

$$\langle \sum_{j=1}^{n} a_j\, \tau_{\xi_j}\ \delta_{\alpha-\beta}, \psi \rangle = \sum_{j=1}^{n} a_j\, \psi\,(\xi_j),\ \ \psi \in \mathcal{H}_{\alpha,\beta}\,,$$

and since $|a_j| = O\left(\xi_j^{\gamma}\right)$ *as* $j \to \infty$, for some $\gamma > 0$, the last sequence converges as $n \to \infty$, for every $\psi \in \mathcal{H}_{\alpha,\beta}$. Therefore

$$\sum_{j=1}^{\infty} a_j\ \tau_{\xi_j}\ \delta_{\alpha-\beta} \in \mathcal{H}'_{\alpha,\beta}\,.$$

Moreover, from (3.3) we deduce that

$$\langle T, \psi \rangle = \langle \sum_{j=1}^{\infty} a_j\, \tau_{\xi_j}\, \delta_{\alpha-\beta},\ h_{\alpha,\beta}\, \psi \rangle$$

$$= \sum_{j=1}^{\infty} a_j\ \ h_{\alpha,\beta}\,(\psi)\,(\xi_j)$$

$$= \langle \sum_{j=1}^{\infty} a_j\,(x\,\xi_j)^{\alpha+\beta} J_{\alpha-\beta}\,(x\,\xi_j),\ \ \psi\,(x) \rangle, \qquad \psi \in \mathcal{H}_{\alpha,\beta}\,.$$

Thus it is established that

$$T = \sum_{j=1}^{\infty} a_j\ (x\,\xi_j)^{\alpha+\beta} J_{\alpha-\beta}\,(x\,\xi_j)\,.$$

It is not hard to see that, if $|a_j| = O\left(|\xi_j|^{-\nu}\right)$ as $j \to \infty$, for each $\nu \in \mathbb{N}$, then by involving well known properties of the Bessel function [27, Sections 5.1, (6) and (7)] for every $b \in \mathbb{N}$ the series

$$\Delta_{\alpha,\beta}^{b}\, T\,(x) = \sum_{j=1}^{\infty} a_j\,\left(-\,\xi_j^2\right)^b\,(x\xi_j)^{\alpha+\beta} J_{\alpha-\beta}\,(x\xi_j)\,,$$

converges uniformly in $x \in I$ and $x^{2\beta-1}\,\Delta^{b}_{\alpha,\beta}\,T$ is bounded on I. Hence,

$$T \in O_{\alpha,\beta,\#} = \bigcup_{m\in\mathbb{Z}} O_{\alpha,\beta,m,\#}\,.$$

Moreover, by proceeding as in the proof of [1, Lemma 2.1] we can conclude that $T \in \bar{O}_{\alpha,\beta,\#}$.

Assume now that $T \in \bar{O}_{\alpha,\beta,\#}$. Let $k \in \mathbb{N}$ and $\psi \in \mathcal{H}_{\alpha,\beta}$. According to [4, (2.1)] and by (3.3) we can write

$$\langle x^{2\beta-1}\,(xh)^{\alpha+\beta}\,J_{\alpha-\beta}\,(xh)\,\Delta^{k}_{\alpha,\beta}\,T(x),\ \phi(x)\rangle$$

$$= \langle\Delta^{k}_{\alpha,\beta}\,T(x), h_{\alpha,\beta}\left(\tau_h\,h_{\alpha,\beta}\,\psi\right)(x)\rangle$$

$$= \langle h'_{\alpha,\beta}T\rangle\,(x),(-x^2)^k\,\tau_h\big(h_{\alpha,\beta}\,\psi\big)\,(x)$$

$$= \sum_{j=1}^{\infty} a_j\,\langle\delta_{\alpha-\beta},\ \tau_{\xi_j}\left(\left((-x^2)^k\tau_h\big(h_{\alpha,\beta}\,\psi\big)\right)\right)\rangle$$

$$= \sum_{j=1}^{\infty} a_j\left(-\xi_j^2\right)^k\tau_{\xi_j}\left(h_{\alpha,\beta}\,\psi\right)(h)$$

$$\int_0^{\infty}(xh)^{\alpha+\beta}\,J_{\alpha-\beta}\,(xh)\left(\Delta^{k}_{\alpha,\beta}\,T\right)(x)\,x^{2\beta-1}\,\psi\,(x)\,dx,\qquad h \in I.$$

Since $x^{2\beta-1}\,\psi(x)\left(\Delta^{k}_{\alpha,\beta}\,T\right)(x)$ is absolutely integrable on I, the Riemann-Lebesgue lemma for the Hankel type transform [20, Section 14.41] leads to

$$\sum_{j=1}^{\infty} a_j\left(-\xi_j^2\right)^k\tau_{\xi_j}\left(h_{\alpha,\beta}\,\psi\right)(h)\ \to 0,\ as\ h\to\infty. \qquad (3.4)$$

We choose a function $\psi \in \mathcal{H}_{\alpha,\beta}$ such that $\psi \not\equiv 0, h_{\alpha,\beta}\,(\psi)\,(x) = 0$ for every $x \geq 1$, and $h_{\alpha,\beta}(\psi) \geq 0$. It is simple to see that such a function ψ can be found.

Then if $x, y \in I$ and $x - y > 1$, we have

$$\tau_x\left(h_{\alpha,\beta}\,\psi\right)(y) = \int_{x-y}^{x+y}\left(h_{\alpha,\beta}\,\psi\right)(z)\,D_{\alpha,\beta}\,(x,y,z)\,dz \qquad (3.5)$$

$$= \int_1^{\infty}\left(h_{\alpha,\beta}\,\psi\right)(z)\,D_{\alpha,\beta}\,(x,y,z)\,dz = 0\,.$$

Moreover, if $x \geq 1/2$ from (2.3) [20, section 13.45] it infers

$$\tau_x\left(h_{\alpha,\beta}\,\psi\right)(x) = \int_0^{2x}\left(h_{\alpha,\beta}\,\psi\right)(z)\,D_{\alpha,\beta}\,(x,x,z)\,dz$$

$$= \frac{x^{4\beta}}{2^{\alpha-5\beta}\,\Gamma(2\alpha)\sqrt{\pi}} \int_0^{2x} z^{-2\beta}\,(4x^2 - z^2)^{-2\beta}\,\left(h_{\alpha,\beta}\,\psi\right)(z)\,dz$$

$$= \frac{1}{2^{\alpha-5\beta}\,\Gamma(2\alpha)\sqrt{\pi}} \int_0^{1} z^{-2\beta}\left(4 - \left(\frac{z}{x}\right)^2\right)^{-2\beta}\left(h_{\alpha,\beta}\,\psi\right)(z)\,dz\,.$$

Hence

$$\tau_x\left(h_{\alpha,\beta}\,\psi\right)(x) \;\to\; \frac{2^{-(\alpha-\beta)}}{\Gamma\,(2\alpha)\,\sqrt{\pi}} \int_0^1 z^{-2\beta}\left(h_{\alpha,\beta}\,\psi\right)(z)\,dz\,, as\ x \;\to\; \infty \qquad (3.6)$$

Note that

$$\int_0^1 z^{-2\beta}\,\left(h_{\alpha,\beta}\,\psi\right)(z)\,dz\;\in I.$$

By virtue of (3.5), for every $l\;\in\;\mathbb{N}$,

$$\sum_{j=1}^{\infty} a_j\,(-1)^k\,\xi_j^{2k}\,\tau_{\xi_j}\,\left(h_{\alpha,\beta}\,\psi\right)(\xi_l) = \;a_l\,(-1)^k\,\xi_l^{2k}\,\tau_{\xi_l}\,\left(h_{\alpha,\beta}\,\psi\right)(\xi_l)\,.$$

Therefore (3.4) and (3.6) imply that $a_l\,\xi_l^{2k} \to 0\ as\ l\;\to\;\infty,$ and the proof is thus completed.

In the following we establish that (HE) is necessary and sufficient in order that $S\;\in\;\bar{O}'_{\alpha,\beta,\#}$ be hypoelliptic in $\mathcal{H}'_{\alpha,\beta}$.

Proposition 3.3: Let $S\;\in\;\bar{O}'_{\alpha,\beta,\#}$. Then S is hypoelliptic in $\mathcal{H}'_{\alpha,\beta}$ if and only if S satisfies (HE).

Proof: Assume firstly that S does not verify (HE). Then, for every $j\;\in\;\mathbb{N}$ there exists $\xi_j\;\in I$ for which

$$\xi_j^{2\beta-1}\left|h'_{\alpha,\beta}\,(S)\left(\xi_j\right)\right|\;\le\;\xi_j^{-j}$$

and $\xi_j - \xi_{j-1}\;>1,\;j = 2,3,\dots\dots\;and\;\xi_1 > 1.$

We now consider $u\;\in\;\mathcal{H}'_{\alpha,\beta}$ such that

$$h'_{\alpha,\beta}\,(u) = \sum_{j=1}^{\infty} \tau_{\xi_j}\,\delta_{\alpha-\beta}\,.$$

According to Proposition 3.2, $u\;\notin\;\bar{O}_{\alpha,\beta,\#}$. Moreover, by invoking [12, Proposition 4.5]

$$h'_{\alpha,\beta}\,(u\#s) =\;x^{2\beta-1}\,h'_{\alpha,\beta}\,(u)\,h'_{\alpha,\beta}\,(S) = \sum_{j=1}^{\infty}\xi_j^{2\beta-1}\;h'_{\alpha,\beta}\,(S)\left(\xi_j\right)\tau_{\xi_j}\,\delta_{\alpha-\beta}\,,$$

and Proposition 3.2 implies that $u\#s \in \bar{O}_{\alpha,\beta,\#}$. Hence S is not hypoelliptic in $\mathcal{H}'_{\alpha,\beta}$. Suppose that S satisfies (HE), and let ψ be a smooth function defined on I such that

$$\psi(x) = \begin{cases} x^{2\alpha}, & for\ 0 < x < C \\ 0, & for\ x \geq C+1, \end{cases}$$

where C is the positive constant that appears in property (HE).

Note that $\psi \in \mathcal{H}_{\alpha,\beta}$.

Also we define

$$P(x) = \begin{cases} 0, & for\ 0 < x \leq C \\ x^{2\alpha} - \phi(x) / \left(x^{2\beta-1} h'_{\alpha,\beta}(S)(x),\ \right) for\ x > C. \end{cases}$$

According to [12, Proposition 4.2] , $x^{2\beta-1} h'_{\alpha,\beta}(S)(x)$ is a multiplier of $\mathcal{H}_{\alpha,\beta}$. Hence as S satisfies (HE) , P is smooth on I. Moreover, $x^{2\beta-1}$ P is a multiplier of $\mathcal{H}_{\alpha,\beta}$. In effect, according to [21] for every $k \in \mathbb{N}$ there exists an $n_k \in \mathbb{N}$ such that

$$(1+x^2)^{-nk} \left(\frac{1}{x} D\right)^k [x^{2\beta-1} h'_{\alpha,\beta}(S)(x)]$$

is bounded on I. Hence since S verifies (HE) by virtue of Theorem in [21], $x^{2\beta-1} P$ is a multiplier of $\mathcal{H}_{\alpha,\beta}$.

We have that

$$x^{2\beta-1} P(x) h'_{\alpha,\beta}(S)(x) = x^{2\alpha} - \phi(x),\ x \in I. \qquad (3.7)$$

By applying the Hankel type transformation to (3.7), it obtains

$$Q \# S = \delta_{\alpha-\beta} - g$$

where $Q = h'_{\alpha,\beta}(P) \in \bar{O}'_{\alpha,\beta,\#}$, [12, *proposition* 4.2], and $\psi = h_{\alpha,\beta}(\psi) \in \mathcal{H}_{\alpha,\beta}$,[25, *Lemma 8.*

Suppose now that $u\#S = v$ where $u \in \mathcal{H}'_{\alpha,\beta}$ and $v \in \bar{O}_{\alpha,\beta,\#}$.

Then, according [12, Proposition 4.7], we can write

$$u = u\#\delta_{\alpha-\beta} = u\#(Q\#S) + u\#g = (u\#S)\#Q + u\#g = v\#Q + u\#g.$$

Proposition 3.1 implies that $v\#Q \in \bar{O}_{\alpha,\beta,\#}$ and [22] leads to $u\#g \in \bar{O}_{\alpha,\beta,\#}$. Thus the hypoellipticity of S is proved.

Thus proof is completed.

Remark 2: Note that by proceeding as in the proof of Proposition 3.3, we can also prove that if $S \in \bar{O}'_{\alpha,\beta,\#}$ and there exist $Q \in \bar{O}'_{\alpha,\beta,\#}$ and $R \in \mathcal{H}_{\alpha,\beta}$ such that

$Q \#S = \delta_{\alpha-\beta} - R$, then S is hypoelliptic in $\mathcal{H}'_{\alpha,\beta}$.

In [24], we introduced for every $m \in Z$ the space $X_{\alpha,\beta,m\#}$ that is formed by all those complex valued and smooth functions ψ defined on I such that for every $k \in \mathbb{N}$,

$$\lambda_k^{\alpha,\beta,m}(\psi) = \operatorname{Sup}_{x \in I} \left| e^{mx} e^{2\beta-1} \Delta_{\alpha,\beta}^k \psi(x) \right| < \infty .$$

It is clear that $X_{\alpha,\beta,m+1,\#}$ is contained in $X_{\alpha,\beta,m,\#}$. By $\chi_{\alpha,\beta,m,\#}$, we denote the closure of $\chi_{\alpha,\beta}$ into $X_{\alpha,\beta,m,\#}$. The space

$$\chi_{\alpha,\beta,\#} = \bigcup_{m \in I} \chi_{\alpha,\beta,m,\#}$$

is endowed with the inductive topology.

Let $S \in \chi'_{\alpha,\beta,\#}$. We say that S (or the Hankel type convolution equation $v\#S = v$) is hypoelliptic in $\chi'_{\alpha,\beta}$ when $v \in \chi_{\alpha,\beta,\#}$ implies that any solution $u \in \chi'_{\alpha,\beta}$ of $u\#S = v \in \chi_{\alpha,\beta,\#}$.

The following property is analogous to the one presented in Proposition 3.1.

Proposition 3.4: If $f \in \chi_{\alpha,\beta,\#}$ and $S \in \chi'_{\alpha,\beta,\#}$, then $f \#S \in \chi_{\alpha,\beta,\#}$.

Proof: We can prove this result in a way similar to Proposition 3.1.

After establishing the following proposition (similar to Proposition 3.2) we will prove that (HE) is also a necessary condition for the hypoelliptic of S in $\chi'_{\alpha,\beta}$.

Proposition 3.5: Let $(\alpha-\beta) \geq 1/2$. Assume that $\xi_j > 2\,\xi_{j-1}$, $j = 2,3, \ldots .,$ and $\xi_1 > 1$. Let $(a_j)_{j=1}^{\infty}$ be a complex sequence such that $|a_j| = O\left(\xi_j^{\gamma}\right)$ as $j \to \infty$ for some $\gamma > 0$. Then

$$\sum_{j=1}^{\infty} a_j\, \tau_{\xi_j}\, \delta_{\alpha-\beta} \in \bar{O}'_{\alpha,\beta} .$$

Moreover, if

$$T = h'_{\alpha,\beta}\left(\sum_{j=1}^{\infty} a_j\, \tau_{\xi_j}\, \delta_{\alpha-\beta}\right), \qquad then \;\; T \in \chi_{\alpha,\beta,\#}$$

if and only if

$$|a_j| = O\left(\xi_j^{-\nu}\right) \, as \, j \to \infty, \text{ for every } \nu \in \mathbb{N} .$$

Proof: Since $Q_{\alpha,\beta} \subset \mathcal{H}_{\alpha,\beta}$ [24] from Proposition 3.2, it is inferred that the series

$$\sum_{j=1}^{\infty} a_j\, \tau_{\xi_j}\, \delta_{\alpha-\beta}$$

converges in $\overline{O}'$ when we consider in $\overline{O}'$ the weak* topology. Then, by [24]

$$T = \sum_{j=1}^{\infty} a_j \left(x\xi_j\right)^{\alpha+\beta} J_{\alpha-\beta}\left(x\xi_j\right) \in \chi'_{\alpha,\beta,\#}\,.$$

Moreover, if $|a_j| = O\left(\xi_j^{-\nu}\right), as\ j \to \infty,$ for each $\nu \in \mathbb{N}$, then it is easy to see that if $T \in \chi_{\alpha,\beta,\#}$. Suppose now that $T \in \chi_{\alpha,\beta,\#}$. Let $k \in \mathbb{N}$ and $\psi \in \chi_{\alpha,\beta}$. We have

$$\sum_{j=1}^{\infty} a_j \left(-\xi_j^2\right)^k \tau_{\xi_j}\left(h_{\alpha,\beta}\,\psi\right)(h)$$

$$= \int_0^\infty (xh)^{\alpha+\beta} J_{\alpha-\beta}(xh)\left(\Delta_{\alpha,\beta}^k\right)(x)\, x^{2\beta-1}\,\psi(x)\,dx \to 0, \qquad (3.8)$$

as $h \to \infty$.

Define $\psi(x) = e^{-x^2} x^{2\alpha}$, $x \in I$. According to (2.10) [7, Section 8.6],

$$h_{\alpha,\beta}(\psi)(y) = \frac{y^{2\alpha}}{2^{3\alpha+\beta}} e^{-y^2/4}, \quad y \in I\,.$$

Hence, since $h_{\alpha,\beta}(\psi) \in \chi_{\alpha,\beta}$, $\psi \in \overline{O}_{\alpha,\beta}$ (See [24]). Note that $h_{\alpha,\beta}(\psi)(y)\, y^{2\beta-1} > 0$ for every $y \in I$.

Let $m \in \mathbb{N}$. We can write

$$\tau_x\left(h_{\alpha,\beta}\,\psi\right)(y) = \int_{|x-y|}^{x+y} D_{\alpha,\beta}(x,y,z)\, h_{\alpha,\beta}(\psi)(z)\,dz \qquad (3.9)$$

$$\le M\,(xy)^{2\alpha}\left(1+|x-y|^2\right)^{-m}, \quad x,y \in I\,.$$

Moreover, for each $x \in I$,

$$\tau_x\left(h_{\alpha,\beta}\,\psi\right)(x) = \int_0^{2x} D_{\alpha,\beta}(x,x,z)\; h_{\alpha,\beta}(\psi)(z)\,dz$$

$$= \frac{x^{4\beta}}{2^{\alpha-5\beta}\Gamma(2\alpha)\sqrt{\pi}} \int_0^{2x} z^{-2\beta}\left((2x)^2 - z^2\right)^{-2\beta} h_{\alpha,\beta}(\psi)(z)\,dz$$

$$= \frac{2^{-(\alpha-\beta)}}{\Gamma(2\alpha)\sqrt{\pi}} \int_0^{2x} z^{-2\beta}\left(1-\left(\frac{z}{2x}\right)^2\right)^{-2\beta} h_{\alpha,\beta}(\psi)(z)\,dz\,.$$

Hence

$$\tau_x\left(h_{\alpha,\beta}\,\psi\right)(x) \to \frac{2^{-(\alpha-\beta)}}{\Gamma(2\alpha)\sqrt{\pi}} \int_0^\infty z^{-2\beta}\left(h_{\alpha,\beta}\,\psi\right)(z)\,dz\,. \qquad (3.10)$$

Let l and $k \in \mathbb{N}$. From (3.9) we deduce that

$$\left|\sum_{j=1}^{\infty} a_j\ (-1)^k\ \xi_j^{2k}\left(\tau_{\xi_j} h_{\alpha,\beta}\ \psi\right)\ (\xi_l)\right|$$

$$\geq\ |a_l|\ \xi_l^{2k}\ \left(\tau_{\xi_j} h_{\alpha,\beta}\ \psi\right)(\xi_l) - \sum_{\substack{j=1\\ j\neq l}} |a_j|\ \xi_j^{2k}\ \left(\tau_{\xi_j} h_{\alpha,\beta}\ \psi\right)\ (\xi_l)$$

$$\geq\ |a_l|\ \xi_l^{2k}\ \left(\tau_{\xi_j} h_{\alpha,\beta}\ \psi\right)(\xi_l) -\ M\ \xi_l^{2k} \sum_{\substack{j=1\\ j\neq l}} |a_j|\ \xi_j^{2k+2\alpha}\ \left(1+\left|\xi_j-\xi_l\right|^2\right)^{-m}$$

$$\geq\ |a_l|\ \xi_l^{2k}\left(\tau_{\xi_j} h_{\alpha,\beta}\ \psi\right)(\xi_l) -\ M\ \ \xi_l^{2k}\ \sum_{\substack{j=1\\ j\neq l}}^{\infty} |a_j|\ \xi_j^{2k+2\alpha}\left|\xi_j-\xi_l\right|^{-m} \qquad (3.11)$$

Since $|a_l| = O\left(\xi_j^{\gamma}\right),\ as\ j\to\infty$ with $\gamma > 0$, one has

$$\sum_{\substack{j=1\\ j\neq l}}^{\infty} a_l\, \xi_j^{2k+2\alpha}\left|\xi_j-\xi_l\right|^{-m}\ \leq M\ \sum_{\substack{j=1\\ j\neq l}}^{\infty} \xi_j^{2k+2\alpha}\left|\xi_j-\xi_l\right|^{-m}. \qquad (3.12)$$

By taking into account that

$$\xi_j - \xi_{j-1}\ \geq 2\ \xi_{j-1} - \xi_{j-1}\ \geq\ 2^{j-1}, \qquad j = 2,3, \ldots,$$

we can obtain

$\left|\xi_j - \xi_l\right|\ \geq\ 2^{l-1}$, for each $j\ \in\ \mathbb{N} - \{l\}$.

Hence, by choosing $m\ \in\ \mathbb{N}$ such that $m\ \geq 2(2k+\gamma+4\alpha+2\beta)$, it follows

$$\sum_{\substack{j=1\\ j\neq l}}^{\infty} \xi_j^{2k+\gamma+2\alpha}\left|\xi_j-\xi_l\right|^{-m}$$

$$\leq \sum_{\substack{j=1\\ j\neq l}}^{\infty} \left|\xi_j-\xi_l\right|^{-1}\ \left|1-\frac{\xi_l}{\xi_j}\right|^{-(2k+\gamma+2\alpha)}\left|\xi_j-\xi_l\right|^{-(2k+\gamma+4\alpha+2\beta)} \qquad (3.13)$$

$$\leq M\ 2^{-l}.$$

By combining (3.11), (3.12) and (3.13), we conclude that $\left|\sum_{\substack{j=1\\ j\neq l}}^{\infty} a_j\ (-1)^k\ \xi_j^{2k}\ \left(\tau_{\xi_j} h_{\alpha,\beta}\ \psi\right)\ (\xi_l)\right|$

$$\geq\ \xi_l^{2k}\left(|a_l|\ \xi_l^{2k+2\beta-1}\tau_{\xi_l}\left(h_{\alpha,\beta}\ \psi\right)(\xi_l) - M2^{-l}\right) \to 0, as\ l\to\infty. \qquad (3.14)$$

Hence, from (3.8), (3.10) and (3.14), we deduce that

$|a_l|\ \xi_l^{2k+2\beta-1}\ \to 0, as\ l\to\infty$. Thus the result is established and hence proof is completed.

The following proposition can be proved as Proposition 3.3.

Proposition 3.6: Let $(\alpha-\beta)\ \geq\ 1/2$ and $S\ \in\ \chi'_{\alpha,\beta,\#}$. If S is hypoelliptic in $\chi'_{\alpha,\beta}$, then S satisfies the property (HE).

Remark 3: Finally we want to remark that, as in $\mathcal{H}'_{\alpha,\beta}$, if $S \in \chi'_{\alpha,\beta,\#}$ and there exist $Q \in \chi'_{\alpha,\beta,\#}$ *and* $R \in \chi_{\alpha,\beta}$ such that

$$Q \# S = \delta_{\alpha-\beta} - R , \tag{3.15}$$

then S is hypoelliptic in $\chi'_{\alpha,\beta}$. However, we do not know how to define $Q \in \chi'_{\alpha,\beta,\#}$ *and* $R \in \chi_{\alpha,\beta}$ satisfying (3.15) when S verifies (HE). We think that the condition (HE) must be replaced by other analogous but stronger conditions than (HE) involving complex values.

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